

Exact Symplectic Form and Hamiltonian on the Kink–Antikink Superposition Ansatz

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Abstract

We compute the exact pullback of the canonical field-theoretic symplectic form and the exact field energy to the four-dimensional parameter space of a kink–antikink superposition ansatz in a general relativistic scalar field theory with degenerate vacua. The diagonal blocks of the symplectic matrix reproduce the free-particle forms $dP_i \wedge da_i$ with the relativistic momenta $P_i = M\gamma_i v_i$. The six off-diagonal interaction components are expressed in closed form through overlap integrals of the static kink profile. The Hamiltonian splits naturally into the sum of the free relativistic energies of two kinks plus an exact interaction potential \mathcal{V} that vanishes exponentially with the separation. In the asymptotic regime of large separation the off-diagonal symplectic couplings are exponentially suppressed, enabling a formal Darboux diagonalisation. We perform the transformation to the canonical momentum variables and discuss the resulting complete classical system as the rigorous starting point for deformation quantisation of the interacting two-kink dynamics. This work opens the path to a rigorous deformation quantisation of the interacting kink–antikink system.

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1 Introduction

The geometric subsystem quantisation programme [1, 2, 3, 4, 5, 6, 7] has provided rigorous quantum theories for single kinks and asymptotically free kink pairs. A central ingredient is the pullback of the canonical symplectic form Ω from the field phase space to a finite-dimensional parameter space. For a single kink this yields $dP \wedge da$; for two widely separated kinks the form factorises into a product of single-kink forms. At arbitrary separation, however, the exact pullback contains off-diagonal interaction terms that couple the parameters.

In this paper we go beyond the symplectic structure and compute also the exact field energy (Hamiltonian) restricted to the same kink–antikink superposition ansatz. The resulting classical system $(\omega^{(2)}, H)$ on the four-dimensional parameter space of positions and velocities is the rigorous field-theoretic counterpart of the heuristic collective-coordinate model. The off-diagonal symplectic couplings and the interaction potential both decay exponentially with the separation, so that in the asymptotic regime the two kinks are independent relativistic particles.

The paper is organised as follows. Section 2 reviews the single-kink results. Section 3 introduces the two-body ansatz and embeds it into the phase space. Section 4 computes the tangent vectors. Section 5 performs the exact pullback of the symplectic form and gives all off-diagonal components. Section 6 derives the exact Hamiltonian in full detail. Section 7 transforms the symplectic form and the Hamiltonian to the natural momentum variables. Section 8 discusses the formal Darboux diagonalisation. Section 9 outlines the steps toward deformation quantisation and summarises the results.

2 Single-kink sector

We consider a relativistic scalar field $\varphi(t, x)$ in $(1+1)$ -dimensional Minkowski space-time with Lagrangian density

$$\mathcal{L} = \frac{1}{2}(\partial_t \varphi)^2 - \frac{1}{2}(\partial_x \varphi)^2 - V(\varphi),$$

where the potential $V : \mathbb{R} \rightarrow \mathbb{R}$ is smooth and possesses at least two degenerate global minima $\varphi_- < \varphi_+$ with $V(\varphi_{\pm}) = 0$ and $V''(\varphi_{\pm}) > 0$. A static kink $f_0(x)$ connecting φ_- to φ_+ satisfies $f_0'' = V'(f_0)$, the first integral $\frac{1}{2}(f_0')^2 = V(f_0)$, and has mass

$$M = \int_{-\infty}^{\infty} (f_0')^2 dx = \int_{\varphi_-}^{\varphi_+} \sqrt{2V(\varphi)} d\varphi.$$

A boosted kink with centre $a \in \mathbb{R}$ and velocity $v \in (-1, 1)$, $\gamma = 1/\sqrt{1-v^2}$, has Cauchy data at $t = 0$:

$$\phi_{a,v}(x) = f_0(\gamma(x-a)), \quad \pi_{a,v}(x) = -\gamma v f_0'(\gamma(x-a)).$$

The pullback of the canonical symplectic form $\Omega = \int (\delta\pi \wedge \delta\phi) dx$ to the translational moduli space $\mathcal{M}_1 = \mathbb{R} \times (-1, 1)$ gives $\omega_1 = dP \wedge dv$ with $P = M\gamma v$ [7].

The antikink is obtained by reflection $f_0(x) = f_0(-x)$, with identical mass and symplectic structure.

3 Two-body ansatz and phase-space embedding

We superimpose a kink and an antikink with independent centres a_1, a_2 and velocities v_1, v_2 . The Cauchy data at $t = 0$ are

$$\begin{aligned}\phi_{a_1, v_1, a_2, v_2}(x) &= f_0(\gamma_1(x - a_1)) + f_0(\gamma_2(a_2 - x)) - \varphi_+, \\ \pi_{a_1, v_1, a_2, v_2}(x) &= -\gamma_1 v_1 f'_0(\gamma_1(x - a_1)) - \gamma_2 v_2 f'_0(\gamma_2(a_2 - x)),\end{aligned}\tag{1}$$

with $\gamma_i = 1/\sqrt{1 - v_i^2}$. The constant $-\varphi_+$ ensures that $\phi(x) \rightarrow \varphi_-$ as $x \rightarrow \pm\infty$ for any finite a_1, a_2 . Thus (ϕ, π) belongs to the affine Sobolev space $\mathcal{A} := (\varphi_- + H^1(\mathbb{R})) \times L^2(\mathbb{R})$. The map

$$\Psi : \mathbb{R}^2 \times (-1, 1)^2 \longrightarrow \mathcal{A}, \quad (a_1, v_1, a_2, v_2) \mapsto (\phi, \pi),$$

is smooth because f_0 is smooth and its derivatives decay exponentially.

Remark 3.1. *The ansatz (1) is not an exact solution of the field equations; it is a point in the full phase space, not in the solution manifold. The pullbacks $\Psi^*\Omega$ and the field energy evaluated on Ψ therefore yield the exact restriction of the canonical structures to a physically motivated finite-dimensional submanifold; they do not describe a subsystem of the exact dynamics.*

4 Tangent vectors

Set $z_1 = \gamma_1(x - a_1)$ and $z_2 = \gamma_2(a_2 - x)$. The derivatives with respect to the parameters are:

$$\begin{aligned}\partial_{a_1}\phi &= -\gamma_1 f'_0(z_1), \\ \partial_{a_1}\pi &= +\gamma_1^2 v_1 f''_0(z_1),\end{aligned}\tag{2}$$

$$\begin{aligned}\partial_{v_1}\phi &= \gamma_1^3 v_1 (x - a_1) f'_0(z_1), \\ \partial_{v_1}\pi &= -\gamma_1 f'_0(z_1) - \gamma_1^3 v_1^2 f'_0(z_1) - \gamma_1^4 v_1^2 (x - a_1) f''_0(z_1), \\ \partial_{a_2}\phi &= -\gamma_2 f'_0(z_2), \\ \partial_{a_2}\pi &= +\gamma_2^2 v_2 f''_0(z_2),\end{aligned}\tag{3}$$

$$\begin{aligned}\partial_{v_2}\phi &= \gamma_2^3 v_2 (a_2 - x) f'_0(z_2), \\ \partial_{v_2}\pi &= -\gamma_2 f'_0(z_2) - \gamma_2^3 v_2^2 f'_0(z_2) - \gamma_2^4 v_2^2 (a_2 - x) f''_0(z_2).\end{aligned}$$

All these vector fields lie in $H^1(\mathbb{R}) \times L^2(\mathbb{R})$ because f'_0, f''_0 decay exponentially.

5 Exact pullback of the symplectic form

The canonical symplectic form on \mathcal{A} is

$$\Omega((\phi_1, \pi_1), (\phi_2, \pi_2)) = \int_{\mathbb{R}} (\pi_1 \phi_2 - \pi_2 \phi_1) dx.$$

The pullback $\omega^{(2)} = \Psi^*\Omega$ is the closed two-form on $\mathbb{R}^2 \times (-1, 1)^2$ with coefficients

$$\omega_{ij}^{(2)} = \int_{\mathbb{R}} (\partial_{q_i} \pi \partial_{q_j} \phi - \partial_{q_j} \pi \partial_{q_i} \phi) dx, \quad q = (a_1, v_1, a_2, v_2).$$

5.1 Diagonal blocks

For the kink parameters (a_1, v_1) , the integrand involves only $f'_0(z_1), f''_0(z_1)$; the antikink part does not contribute because it is not differentiated. The computation is identical to the single-kink case, giving

$$\omega^{(2)}(\partial_{a_1}, \partial_{v_1}) = -M\gamma_1^3, \quad \omega^{(2)}(\partial_{v_1}, \partial_{a_1}) = +M\gamma_1^3.$$

Similarly, the antikink block yields $\omega^{(2)}(\partial_{a_2}, \partial_{v_2}) = -M\gamma_2^3$. Consequently,

$$\omega_{\text{kink}} = dP_1 \wedge da_1, \quad \omega_{\text{antikink}} = dP_2 \wedge da_2, \quad P_i = M\gamma_i v_i.$$

5.2 Overlap integrals

To express the off-diagonal components we introduce the following convergent integrals. Let $f_0^{(0)} = f_0, f_0^{(1)} = f'_0, f_0^{(2)} = f''_0$. For any $m, n \in \{0, 1, 2\}$ define

$$\begin{aligned} I_{mn} &= \int_{\mathbb{R}} f_0^{(m)}(\gamma_1(x - a_1)) f_0^{(n)}(\gamma_2(a_2 - x)) dx, \\ K_{mn} &= \int_{\mathbb{R}} (x - a_1) f_0^{(m)}(\gamma_1(x - a_1)) f_0^{(n)}(\gamma_2(a_2 - x)) dx, \\ L_{mn} &= \int_{\mathbb{R}} (a_2 - x) f_0^{(m)}(\gamma_1(x - a_1)) f_0^{(n)}(\gamma_2(a_2 - x)) dx, \\ M_{11} &= \int_{\mathbb{R}} (x - a_1)(a_2 - x) f'_0(\gamma_1(x - a_1)) f'_0(\gamma_2(a_2 - x)) dx. \end{aligned} \tag{4}$$

All these integrals are finite because f'_0, f''_0 decay exponentially and the polynomial factors do not spoil convergence.

5.3 Explicit off-diagonal components

We now compute each of the six independent off-diagonal components.

5.3.1 Component $\omega^{(2)}(\partial_{a_1}, \partial_{a_2})$

$$\begin{aligned} \partial_{a_1} \pi &= \gamma_1^2 v_1 f''_0(z_1), \quad \partial_{a_2} \phi = -\gamma_2 f'_0(z_2), \\ \partial_{a_2} \pi &= \gamma_2^2 v_2 f''_0(z_2), \quad \partial_{a_1} \phi = -\gamma_1 f'_0(z_1). \end{aligned}$$

Hence

$$\omega^{(2)}(\partial_{a_1}, \partial_{a_2}) = -\gamma_1^2 \gamma_2 v_1 I_{21} + \gamma_1 \gamma_2^2 v_2 I_{12}.$$

5.3.2 Component $\omega^{(2)}(\partial_{a_1}, \partial_{v_2})$

$$\begin{aligned} \partial_{v_2} \phi &= \gamma_2^3 v_2 (a_2 - x) f'_0(z_2), \\ \partial_{v_2} \pi &= -\gamma_2 f'_0(z_2) - \gamma_2^3 v_2^2 f'_0(z_2) - \gamma_2^4 v_2^2 (a_2 - x) f''_0(z_2). \end{aligned}$$

Then

$$\begin{aligned}\omega^{(2)}(\partial_{a_1}, \partial_{v_2}) &= \int_{\mathbb{R}} \left[(\gamma_1^2 v_1 f_0''(z_1)) (\gamma_2^3 v_2 (a_2 - x) f_0'(z_2)) \right. \\ &\quad \left. - (-\gamma_2 f_0'(z_2) - \gamma_2^3 v_2^2 f_0'(z_2) - \gamma_2^4 v_2^2 (a_2 - x) f_0''(z_2)) (-\gamma_1 f_0'(z_1)) \right] dx \\ &= \gamma_1^2 \gamma_2^3 v_1 v_2 L_{21} - \gamma_1 \gamma_2 I_{11} - \gamma_1 \gamma_2^3 v_2^2 I_{11} - \gamma_1 \gamma_2^4 v_2^2 L_{12}.\end{aligned}$$

5.3.3 Component $\omega^{(2)}(\partial_{v_1}, \partial_{a_2})$

By symmetry,

$$\omega^{(2)}(\partial_{v_1}, \partial_{a_2}) = -\gamma_1^3 \gamma_2^2 v_1 v_2 K_{12} + \gamma_1 \gamma_2 I_{11} + \gamma_1^3 \gamma_2 v_1^2 I_{11} + \gamma_1^4 \gamma_2 v_1^2 K_{21}.$$

5.3.4 Component $\omega^{(2)}(\partial_{v_1}, \partial_{v_2})$

$$\begin{aligned}\omega^{(2)}(\partial_{v_1}, \partial_{v_2}) &= \gamma_1^3 \gamma_2^3 v_1 v_2 M_{11} - \gamma_1^3 \gamma_2 v_1 I_{11} - \gamma_1^3 \gamma_2^3 v_1 v_2^2 I_{11} - \gamma_1^3 \gamma_2^4 v_1 v_2^2 L_{12} \\ &\quad - \gamma_1 \gamma_2^3 v_2 I_{11} - \gamma_1^3 \gamma_2^3 v_1^2 v_2 I_{11} - \gamma_1^4 \gamma_2^3 v_1^2 v_2 K_{21}.\end{aligned}$$

The remaining components are obtained from skew-symmetry.

5.4 Complete symplectic matrix in velocity coordinates

The full symplectic matrix in the basis $q = (a_1, v_1, a_2, v_2)$ is

$$\omega^{(2)} = \begin{pmatrix} 0 & -M\gamma_1^3 & C_{13} & C_{14} \\ M\gamma_1^3 & 0 & C_{23} & C_{24} \\ -C_{13} & -C_{23} & 0 & -M\gamma_2^3 \\ -C_{14} & -C_{24} & M\gamma_2^3 & 0 \end{pmatrix},$$

where $C_{13}, C_{14}, C_{23}, C_{24}$ are the off-diagonal coefficients given above. All C_{ij} are smooth functions of (a_1, v_1, a_2, v_2) and decay exponentially as $s = a_2 - a_1 \rightarrow \infty$.

Remark 5.1 (Sign convention). *The canonical symplectic form on the field phase space is taken as $\Omega = \int (\delta\pi \wedge \delta\phi) dx$. With this convention, the diagonal blocks are $dP_i \wedge da_i$ and the Poisson brackets are $\{a_i, P_j\} = \delta_{ij}$. The same convention is used throughout the geometric subsystem quantisation programme [1, 7].*

6 Exact classical Hamiltonian

We now evaluate the field energy functional

$$E[\varphi, \pi] = \frac{1}{2} \int_{\mathbb{R}} (\pi^2 + \varphi_x^2 + 2V(\varphi)) dx$$

on the ansatz (1). With the abbreviations $z_1 = \gamma_1(x - a_1)$, $z_2 = \gamma_2(a_2 - x)$ we have

$$\begin{aligned}\varphi_x &= \gamma_1 f_0'(z_1) - \gamma_2 f_0'(z_2), \\ \pi &= -\gamma_1 v_1 f_0'(z_1) - \gamma_2 v_2 f_0'(z_2).\end{aligned}$$

6.1 Kinetic energy density

$$\begin{aligned}\frac{1}{2}\pi^2 &= \frac{1}{2}\gamma_1^2 v_1^2 (f'_0(z_1))^2 + \frac{1}{2}\gamma_2^2 v_2^2 (f'_0(z_2))^2 + \gamma_1 \gamma_2 v_1 v_2 f'_0(z_1) f'_0(z_2), \\ \frac{1}{2}\varphi_x^2 &= \frac{1}{2}\gamma_1^2 (f'_0(z_1))^2 + \frac{1}{2}\gamma_2^2 (f'_0(z_2))^2 - \gamma_1 \gamma_2 f'_0(z_1) f'_0(z_2).\end{aligned}$$

Adding these,

$$\frac{1}{2}\pi^2 + \frac{1}{2}\varphi_x^2 = \frac{1}{2}\gamma_1^2 (1+v_1^2) (f'_0(z_1))^2 + \frac{1}{2}\gamma_2^2 (1+v_2^2) (f'_0(z_2))^2 + \gamma_1 \gamma_2 (v_1 v_2 - 1) f'_0(z_1) f'_0(z_2).$$

6.2 Single-kink energies

For a single boosted kink the energy density would be

$$\mathcal{H}_i = \frac{1}{2}\gamma_i^2 (1+v_i^2) (f'_0(z_i))^2 + V(f_0(z_i)),$$

and its integral gives the relativistic energy $M\gamma_i$. This follows from the first integral $f_0'^2 = 2V(f_0)$ and the change of variable $dx = dz_i/\gamma_i$.

6.3 Interaction potential

We write the total Hamiltonian as

$$H(a_1, v_1, a_2, v_2) = M\gamma_1 + M\gamma_2 + \mathcal{V}(s, v_1, v_2),$$

where the interaction potential is obtained by subtracting the single-kink contributions from the full field energy:

$$\begin{aligned}\mathcal{V}(s, v_1, v_2) &= \int_{-\infty}^{\infty} \left[\gamma_1 \gamma_2 (v_1 v_2 - 1) f'_0(z_1) f'_0(z_2) \right. \\ &\quad \left. + V(f_0(z_1) + f_0(z_2) - \varphi_+) - V(f_0(z_1)) - V(f_0(z_2)) \right] dx.\end{aligned}$$

All integrals converge because the difference of potentials and the product $f'_0(z_1) f'_0(z_2)$ decay exponentially when x is far from both centres.

6.4 Properties of the interaction potential

- **Exponential decay:** For large separation $s = a_2 - a_1$ the integrand is supported only in a small overlap region, and $\mathcal{V}(s, v_1, v_2) = \mathcal{O}(e^{-\mu s})$ where $\mu = \min\{\sqrt{V''(\varphi_-)}, \sqrt{V''(\varphi_+)}\}$ is the decay rate of the kink profile.
- **Static limit:** Setting $v_1 = v_2 = 0$ (hence $\gamma_i = 1$) eliminates the kinetic cross term, giving the well-known static interaction

$$V_{\text{int}}(s) \equiv \mathcal{V}(s, 0, 0) = \int_{-\infty}^{\infty} \left[V(f_0(x) + f_0(s-x) - \varphi_+) - 2V(f_0(x)) \right] dx.$$

For standard double-well potentials this is attractive and decays as $-V_0 e^{-\mu s}$ with $V_0 > 0$.

- **Non-relativistic reduction:** For low velocities $v_i \ll 1$, expanding $\gamma_i \approx 1 + \frac{1}{2}v_i^2$ and expressing $v_i \approx P_i/M$ (where $P_i = M\gamma_i v_i$ are the canonical momenta) yields

$$H \approx 2M + \frac{P_1^2}{2M} + \frac{P_2^2}{2M} + V_{\text{int}}(s).$$

This is the standard non-relativistic two-body Hamiltonian with reduced mass $\mu = M/2$ and interaction potential $V_{\text{int}}(s)$.

7 Transformation to momentum coordinates

For quantization it is natural to work with the canonical variables (a_i, P_i) . The relativistic momenta are

$$P_i = M\gamma_i v_i, \quad \gamma_i = \sqrt{1 + \frac{P_i^2}{M^2}}, \quad v_i = \frac{P_i}{\sqrt{M^2 + P_i^2}}.$$

The diagonal blocks of the symplectic form become simply $dP_1 \wedge da_1$ and $dP_2 \wedge da_2$. To obtain the off-diagonal components in the new coordinates we must transform the two-form from the (v_1, v_2) basis to the (P_1, P_2) basis. Let J be the Jacobian matrix of the coordinate change; its only non-trivial block is

$$\frac{\partial P_i}{\partial v_i} = M\gamma_i^3, \quad \frac{\partial P_i}{\partial a_j} = 0,$$

so J is diagonal in the velocity-momentum sector. Consequently,

$$\begin{aligned} \omega^{(2)}(\partial_{a_1}, \partial_{P_2}) &= C_{14} \Big/ \frac{\partial P_2}{\partial v_2} = C_{14} / (M\gamma_2^3), \\ \omega^{(2)}(\partial_{P_1}, \partial_{a_2}) &= C_{23} \Big/ \frac{\partial P_1}{\partial v_1} = C_{23} / (M\gamma_1^3), \\ \omega^{(2)}(\partial_{P_1}, \partial_{P_2}) &= C_{24} \Big/ \left(\frac{\partial P_1}{\partial v_1} \frac{\partial P_2}{\partial v_2} \right) = C_{24} / (M^2 \gamma_1^3 \gamma_2^3), \end{aligned}$$

while $\omega^{(2)}(\partial_{a_1}, \partial_{a_2})$ remains unchanged. The transformed off-diagonal coefficients are again exponentially small for large separation s .

The Hamiltonian in the momentum variables is obtained by substituting $v_i = P_i/(M\gamma_i)$ and $\gamma_i = \sqrt{1 + P_i^2/M^2}$ into the expression for H .

8 Formal Darboux diagonalisation

In the asymptotic regime $s \gg 1/\mu$ all off-diagonal symplectic entries are of order $\varepsilon = e^{-\mu s}$. The symplectic form can be written as $\omega^{(2)} = \omega_0 + \varepsilon \omega_1$, where $\omega_0 = dP_1 \wedge da_1 + dP_2 \wedge da_2$ and ω_1 collects the exponentially small corrections. Both ω_0 and $\omega_0 + \varepsilon \omega_1$ are closed and non-degenerate on an open neighbourhood of the asymptotic region. By the formal Darboux theorem (or Moser's lemma for symplectic forms depending on a small parameter), there exists a formal power series

in ε of symplectic diffeomorphisms that transforms $\omega^{(2)}$ to the constant form ω_0 . The coefficients of this transformation can be determined recursively; their explicit construction is not required for the programme that follows—the mere existence of such canonical Darboux coordinates is sufficient.

9 Outlook: toward deformation quantisation

The present paper supplies the complete classical data: the exact symplectic form $\omega^{(2)}$ and the exact Hamiltonian H on the four-dimensional parameter space of the kink–antikink ansatz. Together they define a classical Hamiltonian system that constitutes the rigorous field-theoretic counterpart of the collective-coordinate model.

The existence of formal Darboux coordinates $(\tilde{a}_i, \tilde{P}_i)$ in which the symplectic form is canonical opens the path to a rigorous deformation quantisation. The main steps are:

1. Pull back the classical Hamiltonian $H(a_1, P_1, a_2, P_2)$ to the Darboux coordinates, obtaining $\tilde{H}(\tilde{a}_1, \tilde{P}_1, \tilde{a}_2, \tilde{P}_2)$.
2. Define the Moyal star product in the Darboux coordinates:

$$f \star g = f \exp \left[\frac{i\hbar}{2} \sum_{i=1}^2 \left(\overleftarrow{\partial}_{\tilde{a}_i} \overrightarrow{\partial}_{\tilde{P}_i} - \overleftarrow{\partial}_{\tilde{P}_i} \overrightarrow{\partial}_{\tilde{a}_i} \right) \right] g.$$

3. The star product in the original coordinates is obtained by conjugation with the Darboux map, providing an associative deformation quantisation of the Poisson bracket induced by $\omega^{(2)}$.
4. The quantum Hamiltonian \hat{H} is obtained from \tilde{H} via the Weyl correspondence.

The implementation of this programme will be carried out in a separate work.

10 Conclusion

We have computed the exact pullback of the canonical field-theoretic symplectic form and the exact field energy to the four-dimensional parameter space of a kink–antikink superposition ansatz. The diagonal blocks of the symplectic matrix reproduce the free-particle forms $dP_i \wedge da_i$; the six off-diagonal interaction components have been expressed in closed form through overlap integrals of the static kink profile. The Hamiltonian splits into the sum of two free relativistic energies plus an exponentially decaying interaction potential. The transformation to the natural momentum variables was performed, and the formal Darboux diagonalisation of the exponentially small off-diagonal symplectic couplings was shown to exist.

This work supplies the rigorous classical geometric input necessary for the deformation quantisation of the interacting kink–antikink system and constitutes a key step in the geometric subsystem programme for non-integrable field theories.

11 Application to the double sine–Gordon model

We now specialise the general results of Sections 5 and 6 to the double sine–Gordon (DSG) equation. This model is non-integrable, physically relevant, and supports both elementary kinks and multi-kinks; it therefore provides a non-trivial test of the framework.

11.1 Potential and static kink

The DSG potential is

$$V(\varphi) = 1 - \cos \varphi + \frac{\kappa}{2}(1 - \cos 2\varphi), \quad \kappa > 0. \quad (5)$$

The global minima are $\varphi = 0 \pmod{2\pi}$ and $\varphi = 2\pi n$ for any integer n , with $V(0) = V(2\pi n) = 0$ and $V''(0) = 1 + 2\kappa > 0$. The elementary kink $f_0(x)$ connects $\varphi_- = 0$ to $\varphi_+ = 2\pi$ and satisfies the static ODE $f_0'' = V'(f_0)$ together with the first integral $\frac{1}{2}(f_0')^2 = V(f_0)$. Its mass is

$$M = \int_0^{2\pi} \sqrt{2V(\varphi)} d\varphi = \int_0^{2\pi} \sqrt{2(1 - \cos \varphi + \frac{\kappa}{2}(1 - \cos 2\varphi))} d\varphi.$$

For $\kappa = 0$ this reduces to the sine–Gordon value $M = 8$; for $\kappa > 0$ it is larger and can be computed numerically or expressed in terms of elliptic integrals. The exact closed form of $f_0(x)$ is not known for $\kappa \neq 0$, but the profile decays exponentially with rate $\mu = \sqrt{1 + 2\kappa}$ at both vacua.

11.2 Two-body ansatz

The kink–antikink superposition ansatz (5) becomes

$$\phi_{a_1, v_1, a_2, v_2}(x) = f_0(\gamma_1(x - a_1)) + f_0(\gamma_2(a_2 - x)) - 2\pi, \quad \pi(x) = -\gamma_1 v_1 f_0'(z_1) - \gamma_2 v_2 f_0'(z_2),$$

with $z_1 = \gamma_1(x - a_1)$, $z_2 = \gamma_2(a_2 - x)$. The constant -2π ensures that the field tends to 0 at $x \rightarrow \pm\infty$.

11.3 Symplectic form

The diagonal blocks are again the free-kink forms $dP_i \wedge da_i$ with $P_i = M\gamma_i v_i$. The off-diagonal coefficients are given by the general expressions in Section 5, where the overlap integrals $I_{11}, I_{21}, K_{12}, L_{21}, L_{12}, M_{11}$ are constructed from the DSG kink profile. Although f_0 is not elementary, all these integrals are absolutely convergent because f_0' and f_0'' decay exponentially. They can be evaluated numerically for any given separation and velocities, or studied analytically in the asymptotic regime $s = a_2 - a_1 \gg 1/\mu$.

For $s \gg 1/\mu$, the supports of $f_0'(\gamma_1(x - a_1))$ and $f_0'(\gamma_2(a_2 - x))$ are separated, and every overlap integral is bounded by $Ce^{-\mu s}$. Hence the off-diagonal coefficients C_{ij} are exponentially suppressed, and the symplectic form approaches the free product $dP_1 \wedge da_1 + dP_2 \wedge da_2$. This is a direct consequence of the exponential decay of the

kink profile and the factorisation of the overlap integrals in the asymptotic regime; it is consistent with the physical expectation that widely separated kinks behave independently.

11.4 Exact Hamiltonian

The interaction potential $\mathcal{V}(s, v_1, v_2)$ simplifies because $V(\varphi)$ is now the DSG potential. In the static limit $v_1 = v_2 = 0$,

$$V_{\text{int}}(s) = \int_{-\infty}^{\infty} \left[V(f_0(x) + f_0(s-x) - 2\pi) - 2V(f_0(x)) \right] dx.$$

For large s , the dominant contribution comes from the region where x is close to a_1 and $s-x$ is close to a_2 , so that the two kink tails overlap. Using the asymptotic form of f_0 near the vacuum,

$$f_0(x) \sim C e^{-\mu x} \quad (x \rightarrow \infty), \quad f_0(-x) \sim 2\pi - C e^{\mu x} \quad (x \rightarrow -\infty),$$

one finds, as in the general discussion,

$$V_{\text{int}}(s) \sim -V_0 e^{-\mu s},$$

with $V_0 > 0$; the interaction is attractive. The value of V_0 can be expressed in terms of the coefficient C and the curvature of the potential; for the DSG model it can be computed numerically from the kink profile.

In the non-relativistic limit the Hamiltonian reduces to

$$H \approx 2M + \frac{P_1^2}{2M} + \frac{P_2^2}{2M} - V_0 e^{-\mu s},$$

which is the standard starting point for a quantum mechanical scattering analysis of the DSG kink-antikink pair.

11.5 Relation to earlier DSG results

In [5] the elementary DSG kink was quantised and its translational moduli space was shown to carry the canonical symplectic form $dP \wedge da$. The present paper reproduces this result as the diagonal blocks of the two-body system. In [6] the multi-kink sector (static $2\pi n$ kinks) was quantised using the same pullback method; the single-kink result is again a special case of the general formula when one of the solitons is sent to infinity. Thus the present two-body analysis is fully consistent with the existing geometric quantisation of DSG solitons.

11.6 Numerical evaluation of overlap integrals

Although not required for the theoretical framework, the overlap integrals can be computed numerically by solving the static ODE $f_0'' = V'(f_0)$ with the appropriate boundary conditions and then evaluating the integrals. Such a numerical study would provide the quantitative interaction potential $V_{\text{int}}(s)$ and the off-diagonal symplectic coefficients for any desired separation, enabling a direct comparison with phenomenological collective-coordinate models and with full field-theory simulations of kink-antikink scattering. This is an interesting direction for future work.

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